SL Paper 1

a. Show that $2^n \equiv (-1)^n (\bmod 3)$, where $n \in \mathbb{N}$. [3]

b. Hence show that an integer is divisible by 3 if and only if the difference between the sum of its binary (base 2) digits in even-numbered positions and the sum of its binary digits in odd-numbered positions is divisible by 3. [3]

c. Express the hexadecimal (base 16) number ABBA_{16} in binary. [4]

Consider the recurrence relation $H_{n+1} = 2H_n + 1,\ n \in \mathbb{Z}^+$ where $H_1 = 1$.

a. Find H_2, H_3 and H_4. [2]

b. Conjecture a formula for H_n in terms of n, for $n \in \mathbb{Z}^+$. [1]

c. Prove by mathematical induction that your formula is valid for all $n \in \mathbb{Z}^+$. [5]

Consider the Diophantine equation $7x - 5y = 1,\ x,\ y \in \mathbb{Z}$.

a. Find the general solution to this equation. [3]

b. Hence find the solution with minimum positive value of xy. [2]

c. Find the solution satisfying $xy = 2014$. [3]

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a. A number written in base 5 is 4303. Find this as a number written in base 10. [2]

b. 1000 is a number written in base 10. Find this as a number written in base 7. [5]

The simple connected planar graph J has the following adjacency table.

	A	B	C	D	E	F	G	H
A	–	0	1	1	1	0	0	0
B	0	–	1	1	1	1	1	1
C	1	1	–	1	1	0	1	1
D	1	1	1	–	1	0	0	0
E	1	1	1	1	–	1	1	0
F	0	1	0	0	1	–	1	0
G	0	1	1	0	1	1	–	1
H	0	1	1	0	0	0	1	–

a.i. Without attempting to draw J, verify that *J* satisfies the handshaking lemma; [1]

a.ii. Without attempting to draw J, determine the number of faces in J. [3]

b. The vertices D and G are joined by a single edge to form the graph K. Show that K is not planar. [3]

c.i. Explain why a graph containing a cycle of length three cannot be bipartite. [3]

c.ii. Hence by finding a cycle of length three, show that the complement of K is not bipartite. [2]

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a. Use the Euclidean algorithm to find the greatest common divisor of 74 and 383. [4]

b. Hence find integers *s* and *t* such that 74*s* + 383*t* = 1. [5]

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a. Given that $a \equiv b(\bmod p)$, show that $a^n \equiv b^n(\bmod p)$ for all $n \in \mathbb{Z}^+$. [4]

b. Show that $29^{13} + 13^{29}$ is exactly divisible by 7. [5]

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a. Find the general solution to the Diophantine equation $3x + 5y = 7$. [5]

b. Find the values of x and y satisfying the equation for which x has the smallest positive integer value greater than 50. [2]

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a. Consider the linear congruence $ax \equiv b(\bmod p)$ where $a,\ b \in \mathbb{Z}^+,\ p$ is a prime and $\gcd(a,\ p) = 1$. Using Fermat's little theorem, show that $x \equiv a^{p-2}b(\bmod p)$. [3]

b. **Hence** find the smallest value of x greater than 100 satisfying the linear congruence $3x \equiv 13(\bmod 19)$. [4]

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a. Use the Euclidean algorithm to find $\gcd(162,\ 5982)$. [4]

b. The relation R is defined on $\mathbb{Z}^+$ by nRm if and only if $\gcd(n,\ m) = 2$. [7]

(i) By finding counterexamples show that R is neither reflexive nor transitive.

(ii) Write down the set of solutions of $nR6$.

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a. Express the number 47502 as a product of its prime factors. [2]

b. The positive integers M , N are such that $\gcd(M, N) = 63$ and $lcm(M, N) = 47502$. Given that M is even and $M < N$, find the two possible pairs of values for M , N . [5]

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a. Solve the recurrence relation $u_n = 4u_{n-1} - 4u_{n-2}$ given that $u_0 = u_1 = 1$. [6]

b. Consider v_n which satisfies the recurrence relation $2v_n = 7v_{n-1} - 3v_{n-2}$ subject to the initial conditions $v_0 = v_1 = 1$. [9]

Prove by using strong induction that $v_n = \frac{4}{5}\left(\frac{1}{2}\right)^n + \frac{1}{5}(3)^n$ for $n \in \mathbb{N}$.

Prove that $3k + 2$ and $5k + 3$, $k \in \mathbb{Z}$ are relatively prime.

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a. Prove that the number 14641 is the fourth power of an integer in any base greater than 6. [3]

b. For $a, b \in \mathbb{Z}$ the relation aRb is defined if and only if $\frac{a}{b} = 2^k$, $k \in \mathbb{Z}$. [8]

(i) Prove that R is an equivalence relation.

(ii) List the equivalence classes of R on the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

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a. Using mathematical induction or otherwise, prove that the number $(1020)_n$, that is the number 1020 written with base n , is divisible by 3 for all values of n greater than 2. [8]

b. Explain briefly why the case $n = 2$ has to be excluded. [1]

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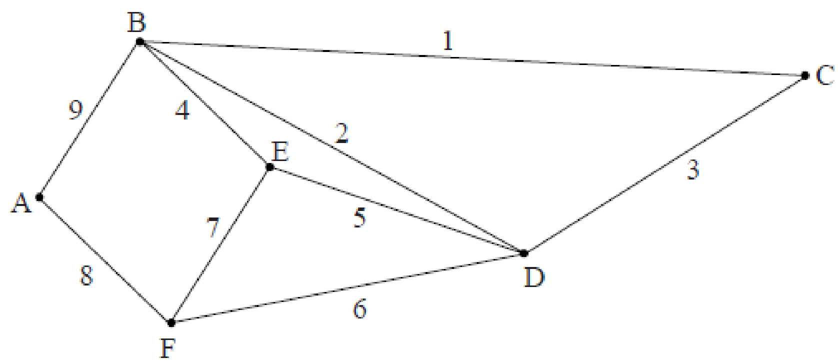
a. Prove that if $\gcd(a, b) = 1$ and $\gcd(a, c) = 1$, then $\gcd(a, bc) = 1$. [5]

b. (i) A simple graph has ***e*** edges and ***v*** vertices, where $v > 2$. Prove that if all the vertices have degree at least ***k*** , then $2e \geq kv$. [6]

(ii) **Hence** prove that every planar graph has at least one vertex of degree less than 6.



The above diagram shows the weighted graph G.

a. Determine whether or not G is bipartite. [2]

b. (i) Write down the adjacency matrix for G. [4]

(ii) Find the number of distinct walks of length 4 beginning and ending at A.

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a. Use the Euclidean Algorithm to show that 275 and 378 are relatively prime. [5]

b. Find the general solution to the diophantine equation $275x + 378y = 1$. [7]

An integer N given in base r, can be expressed in base s in the form

$N = a_0 + a_1 s + a_2 s^2 + a_3 s^3 + \ldots$ where $a_0,\ a_1,\ a_2,\ \ldots \in \{0,\ 1,\ 2,\ \ldots,\ s-1\}$.

a. In base 5 an integer is written 1031. Express this integer in base 10. [2]

b. Given that $N = 365,\ r = 10$ and $s = 7$ find the values of $a_0,\ a_1,\ a_2,\ \ldots$. [2]

c. (i) Given that $N = 899,\ r = 10$ and $s = 12$ find the values of $a_0,\ a_1,\ a_2,\ \ldots$ [3]

(ii) Hence write down the integer in base 12, which is equivalent to 899 in base 10.

d. Show that 121 is always a square number in any base greater than 2. [3]

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a. (i) Use the Euclidean algorithm to find gcd(6750, 144) . [6]

(ii) Express your answer in the form $6750r + 144s$ where r , $s \in \mathbb{Z}$.

b. Consider the base 15 number CBA, where A, B, C represent respectively the digits ten, eleven, twelve. [6]

(i) Write this number in base 10.

(ii) Hence express this number as a product of prime factors, writing the factors in base 4.

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b. (i) Sum the series $\sum_{r=0}^{\infty} x^r$. [11]

(ii) **Hence**, using sigma notation, deduce a series for

(a) $\frac{1}{1+x^2}$;

(b) $\arctan x$;

(c) $\frac{\pi}{6}$.

c. Show that $\sum_{n=1}^{100} n! \equiv 3(\bmod 15)$. [4]

The positive integer N is represented by 4064 in base b and 2612 in base $b+1$.

a. Determine the value of b . [4]

b. Find the representation of N [6]

(i) in base 10;

(ii) in base 12.

Find the positive square root of the base 7 number $(551662)_7$, giving your answer as a base 7 number.

(a) Show that the solution to the linear congruence $ax \equiv b(\bmod p)$, where $a,\ x,\ b,\ p \in \mathbb{Z}^+,\ p$ is prime and a, p are relatively prime, is given by $x \equiv a^{p-2}b(\bmod p)$.

(b) Consider the congruences

$7x \equiv 13(\bmod 19)$

$2x \equiv 1(\bmod 7)$.

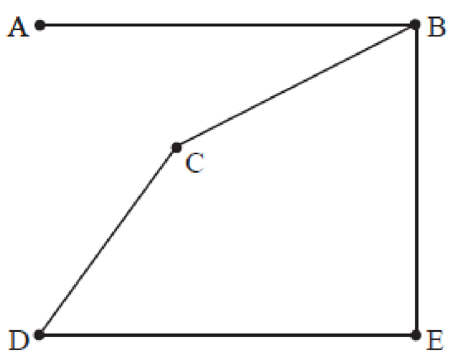
(i) Use the result in (a) to solve the first congruence, giving your answer in the form $x \equiv k(\bmod 19)$ where $1 \leqslant k \leqslant 18$.

(ii) Find the set of integers which satisfy both congruences simultaneously.

Given that $n^2 + 2n + 3 \equiv N(\bmod 8)$, where $n \in \mathbb{Z}^+$ and $0 \le N \le 7$, prove that N can take one of only three possible values.

A sequence $\{u_n\}$ satisfies the recurrence relation $u_{n+2} = 2u_{n+1} - 5u_n$, $n \in \mathbb{N}$. Obtain an expression for u_n in terms of ***n*** given that $u_0 = 0$ and $u_1 = 1$.

The figure below shows the graph ***G*** .



a. (i) Write down the adjacency matrix for G . [5]

(ii) Find the number of walks of length 4 beginning and ending at B.

b. (i) Draw G', the complement of G . [6]

(ii) Write down the degrees of all the vertices of G and all the vertices of G'.

(iii) Hence, or otherwise, determine whether or not G and G' are isomorphic.

A simple graph G is represented by the following adjacency table.

	A	B	C	D	E	F
A	–	1	–	–	1	1
B	1	–	1	–	1	–
C	–	1	–	1	–	–
D	–	–	1	–	1	1
E	1	1	–	1	–	–
F	1	–	–	1	–	–

a. Draw the simple graph G. [1]

b. Explain why G does not contain an Eulerian circuit. [1]

c. Show that G has a Hamiltonian cycle. [2]

d. State whether or not G is planar, giving a reason for your answer. [2]

e. State whether or not the simple graph G is bipartite, giving a reason for your answer. [2]

f. Draw the complement G' of G. [2]